The microstructural foundations for rough volatility: first approach

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Introduction

Volatility is rough !

- We know that : **Volatility is rough** !

- On any asset, using any reasonable volatility proxy/statistical method (realized volatility, realized kernels, uncertainty zones, Garman-Klass, implied volatility, power variations, autocorrelations, Whittle,...), one concludes that volatility is rough.

- It cannot be just coincidence...

- We want to show that typical behaviors of market participants at the high frequency scale naturally lead to rough volatility.

- Our modeling tool : **Hawkes processes**.

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Definition

Hawkes process

- A Hawkes process $(N_t)_{t \geq 0}$ is a self-exciting point process, whose intensity at time t, denoted by λ_t, is of the form

$$\lambda_t = \mu + \sum_{0 < J_i < t} \phi(t - J_i) = \mu + \int_{(0,t)} \phi(t - s) dN_s,$$

where μ is a positive real number, ϕ a regression kernel and the J_i are the points of the process before time t.

- These processes have been introduced in 1971 by Hawkes in the purpose of modeling earthquakes and their aftershocks.

Definition

Order flow and volatility

- Thus, it is nowadays classical to model the order flow (number of trades) thanks to Hawkes processes.

- It is known from financial economics theory (see for example Madhavan, Richardson and Roomans (97)) that the order flow is essentially the same thing as the integrated volatility (variance) if the time scale is large enough :

$$N_t \approx \int_0^t \sigma^2(s)ds.$$

Popularity of Hawkes processes in finance

Two main reasons for the popularity of Hawkes processes

- These processes represent a very natural and tractable extension of Poisson processes. In fact, comparing point processes and conventional time series, Poisson processes are often viewed as the counterpart of iid random variables whereas Hawkes processes play the role of autoregressive processes.

- Another explanation for the appeal of Hawkes processes is that it is often easy to give a convincing interpretation to such modeling. To do so, the branching structure of Hawkes processes is quite helpful.

Hawkes processes as a population model

Poisson cluster representation

- Under the assumption $\|\phi\|_1 < 1$, where $\|\phi\|_1$ denotes the L^1 norm of ϕ, Hawkes processes can be represented as a population process where migrants arrive according to a Poisson process with parameter μ.

- Then each migrant gives birth to children according to a non homogeneous Poisson process with intensity function ϕ, these children also giving birth to children according to the same non homogeneous Poisson process, see Hawkes (74).

- Now consider for example the classical case of buy (or sell) market orders. Then migrants can be seen as exogenous orders whereas children are viewed as orders triggered by other orders.

Stability condition

The condition $\|\phi\|_1 < 1$

- The assumption $\|\phi\|_1 < 1$ is crucial in the study of Hawkes processes.

- If one wants to get a stationary intensity with finite first moment, then the condition $\|\phi\|_1 < 1$ is required (similar condition as for the AR(1) process).

- This condition is also necessary in order to obtain classical ergodic properties for the process.

- For these reasons, this condition is often called stability condition in the Hawkes literature.

$\|\phi\|_1$ in practice

Degree of endogeneity of the market

- From a practical point of view, a lot of interest has been recently devoted to the parameter $\|\phi\|_1$.

- For example, Hardiman, Bercot and Bouchaud (13) and Filimonov and Sornette (12,13) use the branching interpretation of Hawkes processes on midquote data in order to measure the so-called degree of endogeneity of the market, defined by $\|\phi\|_1$.

$\|\phi\|_1$ in practice

Degree of endogeneity of the market

- The parameter $\|\phi\|_1$ corresponds to the average number of children of an individual, $\|\phi\|_1^2$ to the average number of grandchildren of an individual,... Therefore, if we call cluster the descendants of a migrant, then the average size of a cluster is given by $\sum_{k \geq 1} \|\phi\|_1^k = \|\phi\|_1/(1 - \|\phi\|_1)$.

- Thus, the average proportion of endogenously triggered events is $\|\phi\|_1/(1 - \|\phi\|_1)$ divided by $1 + \|\phi\|_1/(1 - \|\phi\|_1)$, which is equal to $\|\phi\|_1$.

$\|\phi\|_1$ in practice

Unstable Hawkes processes

- This branching ratio can be measured using parametric and non-parametric estimation methods for Hawkes processes, see Ogata (78,83) for likelihood based methods and Reynaud-Bouret and Schbath (10) and Al Dayri *et al.* (11) for functional estimators of the function ϕ.

- In Hardiman, Bercot and Bouchaud (13), very stable estimations of $\|\phi\|_1$ are reported for the E-mini S&P futures between 1998 and 2012, the results being systematically close to one.

- This is also the case for Bund and Dax futures in Al Dayri *et al.* (11) and various other assets in Filimonov and Sornette (12).

Aim of our study

Limiting behavior of Hawkes processes

- Our aim is to study the behavior at large time scales of so-called **nearly unstable Hawkes processes**, which correspond to these estimations of $\|\phi\|_1$, close to 1.

- This will give us insights on the properties of the integrated volatility.

- Furthermore, we want to take into account another stylized fact : The function ϕ has typically a power law tail :

$$\phi(x) \underset{x \to +\infty}{\sim} \frac{K}{x^{1+\alpha}},$$

with α of order 0.5-0.7.

- This memory effect is likely due to metaorders splitting.

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The model

Sequence of Hawkes processes

- We consider a sequence of Hawkes processes $(N_t^T)_{t \geq 0}$ indexed by $T \to \infty$ with

$$\lambda_t^T = \mu^T + \int_0^t \phi^T(t-s) dN_s^T.$$

- For some sequence $a_T < 1$, $a_T \to 1$, $K > 0$ and $\alpha \in (0,1)$:

$$\phi^T(t) = a_T \phi(t), \ \alpha x^\alpha (1 - F(x)) \underset{x \to +\infty}{\to} K,$$

with $\|\phi\|_1 = 1$ and

$$F(x) = \int_0^x \phi(s) ds.$$

Non-degenerate limit for nearly unstable Hawkes processes

Martingale process

- Let M^T be the martingale process associated to N^T, that is, for $t \geq 0$,

$$M_t^T = N_t^T - \int_0^t \lambda_s^T ds.$$

- We also set ψ^T the function defined on $\mathbb{R}^+$ by

$$\psi^T(t) = \sum_{k=1}^{\infty} (\phi^T)^{*k}(t).$$

- We can show that

$$\lambda_t^T = \mu^T + \int_0^t \psi^T(t-s)\mu^T ds + \int_0^t \psi^T(t-s)dM_s^T.$$

Non-degenerate limit for nearly unstable Hawkes processes

Rescaling

- We rescale our processes so that they are defined on $[0, 1]$. To do that, we consider for $t \in [0, 1]$

$$\lambda_{tT}^T = \mu^T + \int_0^{tT} \psi^T(Tt - s)\mu^T ds + \int_0^{tT} \psi^T(Tt - s)dM_s^T.$$

- For the scaling in space, a natural multiplicative factor is $(1 - a_T)/\mu^T$. Indeed, in the stationary case,

$$\mathbb{E}[\lambda_t^T] = \mu^T/(1 - \|\phi^T\|_1).$$

Thus, the order of magnitude of the intensity is $\mu^T(1 - a_T)^{-1}$. This is why we define

$$C_t^T = \lambda_{tT}^T(1 - a_T)/\mu^T.$$

Non-degenerate limit for nearly unstable Hawkes processes

Decomposition of C_t^T

- Then we easily get :

$$C_t^T = (1 - a_T) + \int_0^t T(1 - a_T)\psi^T(Ts)ds$$
$$+ \sqrt{\frac{T(1 - a_T)}{\mu^T}} \int_0^t \psi^T(T(t - s))\sqrt{C_s^T}dB_s^T,$$

with

$$B_t^T = \frac{1}{\sqrt{T}} \int_0^{tT} \frac{dM_s^T}{\sqrt{\lambda_s^T}}.$$

Non-degenerate limit for nearly unstable Hawkes processes

The function ψ^T

- The asymptotic behavior of C_t^T is closely linked to that of ψ^T.
- Remark that the function defined for $x \geq 0$ by

$$\rho^T(x) = T \frac{\psi^T(Tx)}{\|\psi^T\|_1}$$

is the density of the random variable

$$X^T = \frac{1}{T} \sum_{i=1}^{I^T} X_i,$$

where the (X_i) are iid random variables with density ϕ and I^T is a geometric random variable with parameter $1 - a_T$.

Non-degenerate limit for nearly unstable Hawkes processes

The function ψ^T

- The Laplace transform of the random variable X^T, denoted by $\widehat{\rho}^T$, satisfies :

$$\widehat{\rho}^T(z) = \frac{\hat{\phi}(\frac{z}{T})}{1 - \frac{a_T}{1-a_T}(\hat{\phi}(\frac{z}{T}) - 1)},$$

where $\hat{\phi}$ denotes the Laplace transform of X_1.

- Due to the assumptions on ϕ, we have

$$\widehat{\phi}(z) = 1 - K\frac{\Gamma(1-\alpha)}{\alpha}z^\alpha + o(z^\alpha).$$

Non-degenerate limit for nearly unstable Hawkes processes

The function ψ^T

- Set $\delta = K\frac{\Gamma(1-\alpha)}{\alpha}$ and $v_T = \delta^{-1}T^\alpha(1 - a_T)$.

- Using that a_T and $\hat{\phi}(\frac{z}{T})$ both tend to 1 as T goes to infinity, $\hat{\rho}^T(z)$ is equivalent to

$$\frac{v_T}{v_T + z^\alpha}.$$

- The function whose Laplace transform is equal to this last quantity is given by

$$v_T x^{\alpha-1} E_{\alpha,\alpha}(-v_T x^\alpha),$$

with $E_{\alpha,\beta}$ the (α, β) Mittag-Leffler function.

Non-degenerate limit for nearly unstable Hawkes processes

Expected limit for C_t^T

- Putting everything together, we can expect (for $\alpha > 1/2$)

$$C_t^T \sim v_T \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-v_T s^\alpha) ds$$
$$+ \gamma_T v_T \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-v_T(t-s)^\alpha)\sqrt{C_s^T} dB_s^T,$$

with

$$\gamma_T = \frac{1}{\sqrt{\mu^T T(1-a_T)}}.$$

- The process B^T can be shown to converge to a Brownian motion B.

Non-degenerate limit for nearly unstable Hawkes processes

Expected limit for C_t^T

- We need that both v_T and γ_T converge to positive constants so we assume :

$$T^{\alpha}(1 - a_T) \to \lambda\delta, \ T^{1-\alpha}\mu^T \to \mu^*\delta^{-1}.$$

- Passing to the limit, we obtain (for $\alpha > 1/2$)

$$C_t^{\infty} \sim \lambda \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^{\alpha}) ds$$
$$+ \sqrt{\frac{\lambda}{\mu^*}} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) \sqrt{C_s^{\infty}} dB_s.$$

Non-degenerate limit for nearly unstable Hawkes processes

Limit theorem

For $\alpha > 1/2$, the sequence of renormalized Hawkes processes converges to some process which is differentiable on $[0, 1]$. Moreover, the law of its derivative V satisfies

$$V_t = F^{\alpha,\lambda}(t) + \frac{1}{\sqrt{\mu^*\lambda}} \int_0^t f^{\alpha,\lambda}(t - s)\sqrt{V_s}dB_s^1,$$

with B^1 a Brownian motion and

$$f^{\alpha,\lambda}(x) = \lambda x^{\alpha-1}E_{\alpha,\alpha}(-\lambda x^\alpha).$$

Non-degenerate limit for nearly unstable Hawkes processes

Rough Heston model

Using fractional integration, we easily get that the equation for V_t on the preceding slide is equivalent to

$$V_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(1-V_s) ds + \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\lambda}{\mu^*}} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s.$$

Now recall Mandelbrot-van-Ness representation :

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

Therefore we have a rough Heston model with $H = \alpha - 1/2$.
Furthermore, for any $\varepsilon > 0$, Y has Hölder regularity $\alpha - 1/2 - \varepsilon$.

Agent based explanation for rough volatility

Microstructural foundations for the RFSV model

- It is clearly established that there is a linear relationship between cumulated order flow and integrated variance.

- Consequently the "derivative" of the order flow corresponds to the spot variance.

- Thus endogeneity of the market together with order splitting lead to a superposition effect which explains (at least partly) the rough nature of the observed volatility.

- Near instability together with a tail index $\alpha \sim 0.6$ correspond to $H \sim 0.1$.